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Mellin transform approach for the solution of coupled systems of fractional differential equations

Salvatore Butera[a,1,*], Mario Di Paola[b]

[a]SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, United Kingdom.
[b]Dipartimento di Ingegneria Civile Ambientale, Aerospaziale e dei Materiali (DICAM), Università degli Studi di Palermo, Viale delle Scienze,Ed.8, 90128 - Palermo, Italy

Abstract

In this paper, the solution of a multi-order, multi-degree-of-freedom fractional differential equation is addressed by using the Mellin integral transform. By taking advantage of a technique that relates the transformed function, in points of the complex plane differing in the value of their real part, the solution is found in the Mellin domain by solving a linear set of algebraic equations. The approximate solution of the differential (or integral) equation is restored, in the time domain, by using the inverse Mellin transform in its discretized form.

Keywords: fractional differential equations, Mellin transform, multi degree of freedom systems

1. Introduction

In the last few decades, the interest of the scientific community towards the fractional calculus experienced an exceptional boost, so that its applications can now be found in a great variety of natural sciences. The powerful of the fractional operators relies in their long memory self-structure, that

*Corresponding author
Email addresses: sb469@hw.ac.uk (Salvatore Butera), mario.dipaola@unipa.it (Mario Di Paola)
[1]Phone number: +44 7462723548

makes them suitable to describe the time evolution of many physical phenomena and, in general, to model the dynamics of complex systems. It has been shown in fact, that fractional differential equations naturally arise once power-type non-local interacting systems, or non-Markovian processes with power-law memory, are considered [1–3]. Relevant examples can be found in electrical circuits [4], in anomalous transport and diffusion processes in complex media [5–8], in material sciences [9–11], in biology [12–14] and biomechanics [15–17], and in many other branches of physics and engineering [18–20]. At the same time, the problem of the solution of these new type of equations came up, and the necessity of a powerful and versatile technique useful to this aim has been object of research. As a result of this effort, various methods are nowadays available in literature. Grünwald-Letnikov [21] or other numerical algorithms [18, 22–25] and Adomian methods [26, 27] are examples, but several others exists [12, 18, 28–31]. Integral transforms as the Laplace and Mellin ones, have been exploited in solving particular classes of fractional differential equations [18]. In [32], the authors presented a general method of solution for single-degree-of-freedom (SDOF) Initial Value Problems (IVP) of fractional order, that makes use of the Mellin transform. The method there proposed, takes advantage of the fact that the result of the inverse Mellin transform is clearly independent on the line of the complex plane along which the integral is carried out, provided it belongs to the so called fundamental strip of the transformation. Since that, exploiting the property of the discretized Mellin transform according to which, in the logarithm temporal scale, it can be seen as a Fourier series, a method that relates the values of the transformed function in different points of the complex plane is derived. This fundamental result allows us to find the solution of the fractional integro-differential equation at hand, in the Mellin domain, by solving a linear set of algebraic equations and, in the time domain, by evaluating the discretized inverse Mellin transform along a proper line of the fundamental strip.

In this paper we show that such a method, developed for a SDOF system, can be straightforwardly generalized for the solution of a general linear multi-order, multi-degree-of-freedom (MDOF) fractional (or integral) differential equation. Again, the method is versatile and easy to implement in computer routines.

The paper is organised as follows: in the next section, the basic concepts of the Mellin transform and the application of the method to a SDOF system is resumed, along with an illustrative application. In section 3, the generalisation to a multi-order, MDOF system is presented; in section 4, the application of the method to a relevant example of a structural system is illustrated.

2. Mellin transform and SDOF systems

In this section, we recall the definition of the Mellin integral transform and we outline the ideas underlying the method developed in [32] for a SDOF. We illustrate it by giving an exemplifying application for the solution of the fractional Kelvin-Voigt equation, that models the rheological properties of a viscoelastic material, in which the classical dashpot is substituted by a spring-pot, characterized by a constitutive law of fractional order.

The Mellin transform of complex order $\gamma = \rho + i\,\eta$, of a function $x(t)$, defined in the time domain $t \geq 0$, is given as:

$$X\left(\gamma\right) = \mathcal{M}\left\{x(t);\gamma\right\} \equiv \int_0^{\infty} t^{\gamma-1}\, x(t)\, dt \tag{1}$$

along with its inverse transform

$$x\left(t\right) = \mathcal{M}^{-1}\left\{X(\gamma);t\right\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\gamma)\, t^{-\gamma}\, d\eta; \qquad (t>0) \tag{2}$$

Eqs.(1) and (2) exist provided ρ belongs to the strip of the complex plane $-p < \rho < -q$, known as fundamental strip of the Mellin transform, in which the transformed function is holomorphic. The limits of the fundamental strips are related to the asymptotic behaviour of the function at hand, for $t \to 0$ and $t \to \infty$. In particular:

$$x(t) \sim t^p \quad (t \to 0)\ ; \quad x(t) \sim t^q \quad (t \to \infty) \tag{3}$$

The method is formulated by taking advantage of a discretized version of the inverse transform (2), that we introduce as:

$$x(t) \simeq \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} X(\gamma_k) t^{-\gamma_k} \tag{4}$$

where $\gamma_k = \rho + i\, k\Delta\eta$ and $\bar{\eta} = m\Delta\eta$ is a properly selected cutoff value for the integral along the imaginary axes. Taking advantage of the property $X(\gamma_k) = X^*(\gamma_{-k})$, we can rewrite eq.(4) as

$$x(t) \simeq t^{-\rho} \left\{ \frac{A_0}{2b} + \frac{1}{b} \sum_{k=1}^{m} \left[A_k \cos\left(\frac{k\pi}{b} \ln t\right) + B_k \sin\left(\frac{k\pi}{b} \ln t\right) \right] \right\} \tag{5}$$

with $b = \pi/\Delta\eta$ and having indicated $A_k = \mathrm{Re}\left[X(\gamma_k)\right]$ and $B_k = \mathrm{Im}\left[X(\gamma_k)\right]$. From eq.(5) it is evident that the inverse Mellin transform may be seen, in the logarithm scale, as a Fourier series. Since the same function $x(t)$ is restored whichever is the value of ρ belonging to the fundamental strip, according to eq.(4), we can pose

$$\frac{\Delta\eta}{2\pi} \sum_{s=-m}^{m} X(\gamma_{1s}) t^{-\gamma_{1s}} \simeq \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} X(\gamma_{2k}) t^{-\gamma_{2k}} \tag{6}$$

provided $\rho_1 = \mathrm{Re}\left[\gamma_1\right]$ and $\rho_2 = \mathrm{Re}\left[\gamma_2\right]$, belong to the fundamental strip. Eq.(6) allows us to relate the Mellin transform $X(\gamma)$, in points of the complex plane differing in their real part. To show that, let us suppose the values $X(\gamma_{2k})$ known, and that we want to evaluate the $X(\gamma_{1s})$ $(k, s = -m, ..., m)$, with $\rho_1 < \rho_2$. Indicating $\delta = \rho_2 - \rho_1$, and multiplying both sides of eq.(6) for the factor $t^{-1/2}$, we get

$$t^{-\frac{1}{2}} \sum_{s=-m}^{m} X(\gamma_{1s}) \exp\left(-i \frac{s\,\pi}{b} \ln t\right) \simeq t^{-\left(\delta + \frac{1}{2}\right)} \sum_{k=-m}^{m} X(\gamma_{2k}) \exp\left(-i \frac{k\,\pi}{b} \ln t\right) \tag{7}$$

We then minimize, respect to $X^*(\gamma_{1k})$, the squared modulus of the difference between the two sides of eq.(7), integrated over a proper time interval, that is

$$\int_{t_1}^{t_2} \frac{1}{t} \left[\sum_{s=-m}^{m} X(\gamma_{1s}) \exp\left(-i\frac{s\pi}{b}\ln t\right) - t^{-\delta} \sum_{k=-m}^{m} X(\gamma_{2k}) \exp\left(-i\frac{k\pi}{b}\ln t\right)\right] \times \left[\text{c.c.}\right] dt = \min_{X^*(\gamma_{1s})} \tag{8}$$

where [c.c.] stands for complex conjugate. In order overcome the singularity in $t = 0$, we choose the lower limit of the integral $t_1 = e^{-b}$ and we consider $t_2 = e^b$ as upper limit. In this way, the overlapping of the response, evaluated along the two different lines $\rho = \rho_1$ and $\rho = \rho_2$ of the fundamental strip, is guaranteed in a wide time interval. Making now the change of variable

$$\ln t = \xi\,; \quad \frac{dt}{t} = d\xi\,; \quad \ln t_1 = -b\,; \quad \ln t_2 = b \tag{9}$$

performing variations of eq.(8) respect to $X^*(\gamma_{1s})$, and taking advantage of the orthogonality of the exponentials $\exp\left(-i\,\frac{s\pi}{b}\xi\right)$ on the interval $\xi = [-b, b]$, we get the following relation between $X(\gamma_{1s})$ and $X(\gamma_{2k})$:

$$X\left(\gamma_{1s}\right) = \frac{1}{2b} \sum_{k=-m}^{m} X(\gamma_{2k}) a_{sk}(\delta) \tag{10}$$

where

$$a_{sk}(\delta) = \int_{-b}^{b} \exp\left(-\left(\delta - i\pi\frac{s-k}{b}\right)\xi\right) = 2b\frac{\sin\left((s-k)\pi + ib\delta\right)}{(s-k)\pi + ib\delta} \tag{11}$$

Thanks to eq.(11), we are able to solve, in the Mellin domain, whichever muti-order linear fractional integro-differential equation, by simply solving a linear set of algebraic equations.

Before addressing the more general case of a MDOF problem, which is the object of the following sections, we briefly outline here the application of the method for the solution of a relevant SDOF physical problem, that is the response, in term of strain, of a sample of viscoelastic material enforced by a given stress history. We model the material through a fractional Kelvin-

Voigt element, whose constitutive law is given by

$$f(t) = c_0 x(t) + c_\alpha \left({}^C\mathbf{D}^\alpha_{0+} x\right)(t) \tag{12}$$

where ${}^C\mathbf{D}^\alpha_{0+}$ is the Caputo's fractional differential operator of order $0 < \alpha < 1$ and $f(t)$ is the stress history applied to the sample. Assuming the system quiescent for $t < 0$, the Cauchy problem that is to be solved, reads as

$$\begin{cases} f(t) = c_0 x(t) + c_\alpha \left({}^C\mathbf{D}^\alpha_{0+} x\right)(t) & \text{(13a)} \\ x(0) = 0 & \text{(13b)} \end{cases}$$

By Melling transforming eq.(13a), we get the corresponding equation in the complex plane, that reads as:

$$c_\alpha \sum_{k=0}^{n-1} \frac{\Gamma(\alpha-\gamma)}{\Gamma(1-\gamma)} \left[x(t)\, t^{\gamma-\alpha}\right]_0^\infty + c_\alpha \frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma)} X(\gamma-\alpha) + c_0 X(\gamma) = F(\gamma) \tag{14}$$

being $F(\gamma) = \mathcal{M}\{f(t);\gamma\}$ the Mellin transform of the forcing action. In the hypothesis of stable system c_0, $c_\alpha > 0$, and assuming: i) $f(t) \equiv 0$ from a time instant $t = \bar{t}$, condition that doesn't represent a limitation since, because of the causal properties of the system at hand, the response at $t = \bar{t}$ is only determined by the past stress history, ii) $\rho < \alpha$, then eq.(14) reduces to

$$c_\alpha C(\gamma,\alpha) X(\gamma-\alpha) + c_0 X(\gamma) = F(\gamma) \tag{15}$$

having defined $C(\gamma,\alpha) = \frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma)}$. It is clear that the solution can't be sought directly from eq.(15) because values of the transformed function in different points of the complex plane are involved. However, by taking advantage of eq.(10), we are in the position to rewrite eq.(15), in terms of only the values that the transformed function assumes along the same line of the fundamental strip. Evaluating the resulting equation in the points $\gamma_k = \rho + i\, k\Delta\eta$ $(k = -m, ..., m)$, we obtain a system of $2m+1$ equations in the $2m+1$ unknown values $X(\gamma_k)$:

$$\mathbf{MX} = \mathbf{F} \tag{16}$$

in which we have defined

$$M_{kj} = \frac{1}{2b}\left(c_\alpha C_k^\alpha a_{k-m-1,j-m-1}(\alpha) + c_0 \delta_{kj}\right) \tag{17a}$$

$$X_k = X(\rho + i(k-m-1)\Delta\eta) \tag{17b}$$

$$F_k = F(\rho + i(k-m-1)\Delta\eta) \tag{17c}$$

and $k, j = 1 \div 2m{+}1$, $C_k^\alpha = C\left(\rho + i(k-m-1)\Delta\eta, \alpha\right)$. Once solved eq.(16), the approximate solution, in the time domain, can be restored by using the discretized inverse Mellin integral (4). Such integral is meaningful provided ρ belongs to the fundamental strip of the transformed function $X(\gamma)$, that a priori is not known. However, because of the initial condition (13b), the lower limit is at least equal to -1 while, by comparison with other similar situations, we can guess the upper limit to be infinite [32].

3. MDOF systems

In this section we show that the method outlined above for the solution of a multi-order SDOF fractional differential equation, can be straightforwardly generalized to the more general case of a MDOF system. An attempt to address the problem can be found in [33] where, defining in a proper way the state vector of the system at hand, the problem can be reduced, in the modal space, to a mutually independent set of fractional differential equations. Aim of this section is to show that this step can be overcame, and that the problem can be directly addressed in the relative physical space. We first illustrate the method by dealing with the most general linear multi-order, MDOF fractional differential equation with constant coefficients. Then, in the next section, we give a relevant application for the solution of the dynamics of a structural system, enforced by a time dependent action, in which the presence of viscoelastic dampers cause the appearance of differential operators of not integer order in the equation of motion.

Let us assume that we are involved in solving the following M-degree of freedom Cauchy problem of order $n-1<\alpha_N<n$, with $n\in\mathbb{N}$:

$$\begin{cases} \mathbf{A}_1\left({}^C\mathbf{D}_{0^+}^{\alpha_1}\mathbf{x}\right)(t)+\mathbf{A}_2\left({}^C\mathbf{D}_{0^+}^{\alpha_2}\mathbf{x}\right)(t)+...+\mathbf{A}_N\left({}^C\mathbf{D}_{0^+}^{\alpha_N}\mathbf{x}\right)(t)=\mathbf{f}(t) & \text{(18a)}\\ \mathbf{x}(0)=\mathbf{0}\,; \quad ... \quad \mathbf{x}^{(n-1)}(0)=\mathbf{0} & \text{(18b)}\end{cases}$$

with $\alpha_1<\alpha_2<...<\alpha_N$, $\mathbf{x}(t)^T=[x_1(t),x_2(t),...,x_M(t)]$ and $\mathbf{f}(t)^T=[f_1(t),f_2(t),...,f_M(t)]$. In the hypothesis of stable systems ($\mathbf{A}_j$ positive definite for $j=1,...,N$), assuming (because of causality and so without affecting generality) the forcing action $\mathbf{f}(t)$ different from zero in the finite time interval $[0,\bar{t}]$ and zero otherwise, and posing $\rho<\{\alpha\}$ (being $\{\alpha\}$ the not integer part of α), the Mellin transform of eq.(18), reads as:

$$C(\gamma,\alpha_1)\,\mathbf{A}_1\mathbf{X}(\gamma-\alpha_1)+...+C(\gamma,\alpha_N)\,\mathbf{A}_N\mathbf{X}(\gamma-\alpha_N)=\mathbf{F}(\gamma) \tag{19}$$

with $\mathbf{X}(\gamma)^T=[X_1(\gamma),X_2(\gamma),...,X_M(\gamma)]$, $\mathbf{F}(\gamma)^T=[F_1(\gamma),F_2(\gamma),...,F_M(\gamma)]$, and $C(\gamma,\alpha_j)$ as defined in the previous section. In a more compact form we can also write:

$$\sum_{j=1}^{N}C(\gamma,\alpha_j)\,\mathbf{A}_j\mathbf{X}(\gamma-\alpha_j)=\mathbf{F}(\gamma) \tag{20}$$

As for the SDOF case, the solution of the problem cannot be pursued directly from eq.(20), because the values $\mathbf{X}(\gamma-\alpha_j)$ $(j=1,...,N)$ are involved in the same equation. However, we know from eq.(10), that the values of the transformed function, in points belonging to different line of the fundamental strip, are related as

$$\mathbf{X}(\gamma_s-\alpha_j)=\frac{1}{2b}\sum_{k=-m}^{m}\mathbf{X}(\gamma_k)a_{sk}(\alpha_j) \tag{21}$$

with $s=-m\div m$ and remembering that m is related to the cut-off value $\bar{\eta}$ of the inverse Mellin integral as $m\Delta\eta=\bar{\eta}$. Evaluating eq.(20) in the point γ_s, and exploiting eq.(21), we end up with the following equation:

$$\sum_{k=-m}^{m}\mathbf{M}(s,k)\mathbf{X}(\gamma_k)=\mathbf{F}(\gamma_s) \tag{22}$$

where we defined

$$\mathbf{M}(s,k) = \frac{1}{2b}\sum_{j=1}^{N} C\left(\gamma_s,\alpha_j\right) a_{sk}(\alpha_j)\mathbf{A}_j \tag{23}$$

By evaluating eq.(22) for $s=-m,...,m$, we obtain a linear set of $2m+1$ algebraic equations in the $2m+1$ vectorial unknown $\mathbf{X}(\gamma_s)$. By defining the $(2m+1)\cdot M$ dimensional super-vectors $\boldsymbol{\chi}^T = \left[\mathbf{X}^T_{-m},...,\mathbf{X}^T_m\right]$, $\boldsymbol{\phi}^T = \left[\mathbf{F}^T_{-m},...,\mathbf{F}^T_m\right]$ and the $(2m+1)\cdot M \times (2m+1)\cdot M$ block matrix $\boldsymbol{\mu}$, composing the $\mathbf{M}(s,k)$ sub-matrices as follows:

$$\boldsymbol{\mu} = \begin{pmatrix} \mathbf{M}(-m,-m) & \mathbf{M}(-m,-m+1) & \cdots & \mathbf{M}(-m,m) \\ \mathbf{M}(-m+1,-m) & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{M}(m,-m) & \dots & \dots & \mathbf{M}(m,m) \end{pmatrix}$$

we can rewrite eq.(22) in the more compact form:

$$\boldsymbol{\mu}\,\boldsymbol{\chi} = \boldsymbol{\phi} \tag{24}$$

Once found the approximate solution of the problem in the Mellin domain by solving eq.(24), we can restore the sought solution in the time domain, by evaluating the inverse transform (4).

4. Applications

In order to show the versatility and the powerful of the method, we present here its application for the solution of the dynamics of a structural system, equipped with viscoelastic dampers, subject to the action of a time-dependent load. In order to validate the method, we choose the structural parameters in such a way that the problem is diagonal in its modal space, and so easily solvable by applying the method to the resulting uncoupled SDOF fractional equations, as in the previous section.

The equation of motion of a M-degree-of-freedom structure, equipped

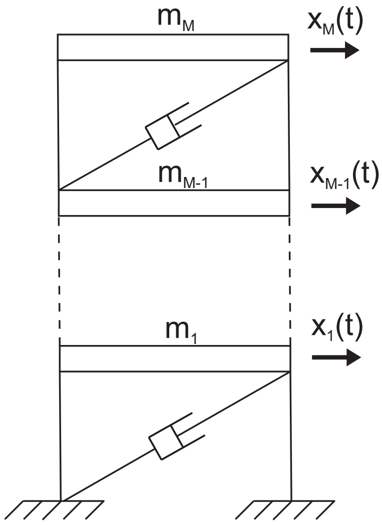


Figure 1: Schematic model of a multi-degree-of-freedom structural system, passively protected with viscoelastic dampers.

with viscoelastic dampers (see fig.1), reads as:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{C}_\alpha \left({}^{C}\mathbf{D}^{\alpha}_{0^+}\mathbf{x}\right)(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{25}$$

in which $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ and $\mathbf{C}_\alpha$ are respectively the damping and fractional damping matrices, and $\mathbf{K}$ represent the stiffness matrix of the structure. Assuming the system quiescent for $t < 0$, in the same hypothesis as above, the corresponding equation in the Mellin domains reads:

$$\begin{aligned} C(\gamma, 2)\,\mathbf{M}\mathbf{X}(\gamma - 2) + C(\gamma, 1)\,\mathbf{C}\mathbf{X}(\gamma - 1) \\ + C(\gamma, \alpha)\,\mathbf{C}_\alpha\mathbf{X}(\gamma - \alpha) + \mathbf{K}\mathbf{X}(\gamma) = \mathbf{F}(\gamma) \end{aligned} \tag{26}$$

Evaluating eq.(26) in the point $\gamma_s = \rho + is\Delta\eta$ of the fundamental strip, we can write it in terms of the only values $X(\gamma_k)$ of the transformed function, by making use of eq.(21), as follows:

$$\sum_{k=-m}^{m} \mathbf{M}(s, k)\,\mathbf{X}(\gamma_k) = \mathbf{F}(\gamma_s) \tag{27}$$

with

$$\mathbf{M}(s,k) = \frac{1}{2b}\left[C(\gamma_s,2)\, a_{sk}(2)\mathbf{M} + C(\gamma_s,1)\, a_{sk}(1)\mathbf{C}\right.$$
$$\left. + C(\gamma_s,\alpha)\, a_{sk}(\alpha)\mathbf{C}_\alpha\right] + \delta_{sk}\mathbf{K} \quad (28)$$

Evaluating eq.(27) for $s = -m, ..., m$, we get the solving algebraic set of equations, as obtained in (24).

In order to validate the method, we consider the simpler case in which $\mathbf{C} = \lambda_1 \mathbf{K}$, $\mathbf{C}_\alpha = \lambda_\alpha \mathbf{K}$, condition necessary for the system to be diagonal in its modal space. Under these conditions, defining the dynamical matrix $\mathbf{D} = \mathbf{M}^{-1}\mathbf{K}$, the equation of motion of the system takes the form:

$$\ddot{\mathbf{x}}(t) + \mathbf{D}\left(\lambda_1 \dot{\mathbf{x}}(t) + \lambda_\alpha \left({}^{C}\mathbf{D}^{\alpha}_{0+}\mathbf{x}\right)(t) + \mathbf{x}(t)\right) = \mathbf{g}(t) \quad (29)$$

with $\mathbf{g}(t) = \mathbf{M}^{-1}\mathbf{f}(t)$. Let us rewrite now the physical position vector $\mathbf{x}(t)$, in the basis composed by the normalized eigenvector $\boldsymbol{\phi}_i$ $(i = 1, ..., N)$ of the dynamical matrix $\mathbf{D}$. Labelling $\mathbf{y}(t)$ the modal vector and $\mathbf{\Phi}$ the transformation matrix between the physical and the modal space, whose column are the eigenvectors $\boldsymbol{\phi}_i$, we can write:

$$\mathbf{x}(t) = \mathbf{\Phi}\mathbf{y}(t) \quad (30)$$

By inserting eq.(30) into eq.(29), and pre-multiplying both sides by $\mathbf{\Phi}^T$, we get

$$\ddot{\mathbf{y}}(t) + \mathbf{U}_D\left(\lambda_1 \dot{\mathbf{y}}(t) + \lambda_\alpha \left({}^{C}\mathbf{D}^{\alpha}_{0+}\mathbf{y}\right)(t) + \mathbf{y}(t)\right) = \mathbf{h}(t) \quad (31)$$

having defined $\mathbf{U}_D = \mathbf{\Phi}^T\mathbf{D}\mathbf{\Phi} = \mathrm{diag}\{\epsilon_1, ..., \epsilon_M\}$ (with ϵ_j the j-th eigenvalue of $\mathbf{D}$) and $\mathbf{h}(t) = \mathbf{\Phi}^T\mathbf{g}(\mathbf{t})$ the forcing action in the modal space. The eq.(30) is a set of independent fractional differential equations, having the form:

$$\ddot{y}_j(t) + \epsilon_j\left(\lambda_1 \dot{y}_j(t) + \lambda_\alpha \left({}^{C}\mathbf{D}^{\alpha}_{0+}y_j\right)(t) + y_j(t)\right) = h_j(t) \quad (32)$$

Once the solution of eq.(32) is found for $j = 1, ..., M$, the response in the physical space is restored by eq.(30).

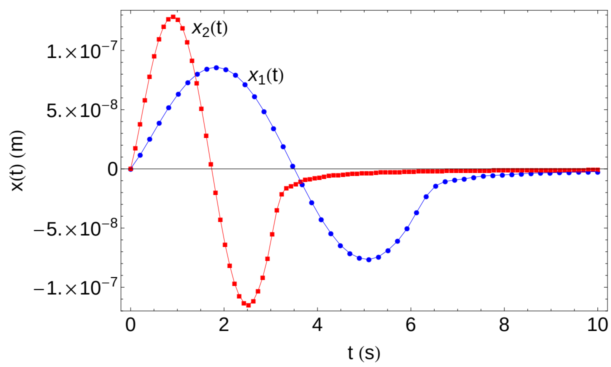
We consider as first example a 2-DOF structure, with the value $\alpha = 0.2$ for the order of the fractional derivative. For simplicity we assume significant only the stiffness due to the main structure, disregarding the stiffness of the viscoelastic dampers. Indicating with $K = 6EI/\ell^2$ the stiffness of each ℓ long beam, where E is the Young's modulus of the material and I the moment of inertia of the beam cross section, and with m_1 and m_2 the values of the two masses, then the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$, read as:

$$\mathbf{K} = \begin{pmatrix} 2K & -K \\ -K & K \end{pmatrix}; \qquad \mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{33}$$

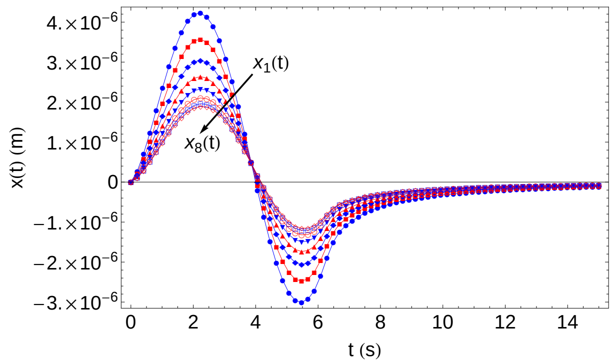
In fig.2(a) is reported the solution obtained from eq.(27), compared to the one obtained by first solving the system in its modal space, when the following forcing function acts on every degree of freedom of the system:

$$\mathbf{f}(t) = \begin{cases} \sin t & 0 \le t \le 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

The values of the parameters $\lambda_1 = 1$, $\lambda_\alpha = 3$, and the values $m_1 = m_2 = 10^4\,\mathrm{Kg}$ and $E = 2 \times 10^{10}\,\mathrm{N/m^2}$ for the physical quantities defined above have been used. We considered a length $\ell = 4\,\mathrm{m}$ for the beams, and a square $0.3 \times 0.3\,\mathrm{m}$ cross section. Both the solutions have been calculated by evaluating the discretized inverse Mellin transform on the line $\rho = 0.5$ of the complex plane, considering a cut-off $\bar{\eta} = 50$ and a sampling step $\Delta\eta = 0.5$.



(a) 2-DOF



(b) 8-DOF

Figure 2: Response of a structural system, equipped with viscoelastic dampers and enforced by the time dependent load given in eq.(34), applied to the whole structure in the 2-DOF case and only to the first mass (m_1) in the 8-DOF case. Comparison between the solution obtained from eq.(27) (dotted line) and from the modal analysis eq.(31) (continuous line).

In fig.2(b) is reported the solution for a 8-DOF system, enforced by a load with a time dependence as in eq.(34), and acting only on the first mass of the structure. We used the value $\alpha = 0.5$ for the order of the fractional derivative, $E = 2 \times 10^8 \mathrm{N/m}^2$ for the Young's modulus of the material and the same values for the λ_1, λ_α parameters and for the others physical and geometrical quantities of the system, as for the 2-DOF case.

5. Conclusions

We generalized the method of solution for single-degree-of freedom (SDOF) fractional differential equations, presented by the authors in [32], to the more general case of multi-degree-of-freedom (MDOF) systems. By taking advantage of the theory of the Mellin transform in the complex plane, the method allows us to solve the most general multi-order, MDOF fractional (integro-)differential equation with constant coefficients, regardless is the number and the order of the differential operators. An approximate solution is found in the Mellin domain by solving a linear set of algebraic equations, and the corresponding solution in the physical domain is restored by using a discretized version of the inverse Mellin transform. The powerful and the versatility of the method have been proved by mean of an illustrative application for the solution of the dynamics of two examples of passively protected structural systems enforced by a time dependent load, in which the presence of viscoelastic dampers causes the appearance of non integer order operators in the equations of motion. Such solutions have been verified by a comparison with the ones obtained by analysing the system in its modal space, in which the relative dynamics reduce to a set of decouples fractional differential equations.

The method revealed robust, computationally efficient, and easily implementable for any order and number of the fractional operators present in the MDOF fractional differential equations.

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